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RANKING THE NODES IN DIRECTED AND WEIGHTED DIRECTED GRAPHS

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Ranking the Nodes in Directed and Weighted Directed Graphs[‡]

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Abstract

A *directed graph* (N, D) can be interpreted as a hierarchical organization of the nodes in N in which the arcs in D represent *dominance relations* between these nodes. A *ranking procedure* is a mapping that assigns to every digraph a ranking or ordering of the nodes such that nodes that are ranked 'higher' are more important in the organization than nodes that are ranked 'lower'. In this paper we axiomatize the ranking procedure that is based on the *score measure*. In this ranking procedure a node is 'better' the more other nodes it dominates.

A *weighted directed graph* can be interpreted as a dominance structure whose dominance relations are assigned weights representing the 'importance' of these relations. We extend the ranking by score measure to the class of weighted directed graphs and give an axiomatic characterization of it.

1 Introduction

A *directed graph* or *digraph* is a pair (N, D) where N is a set of nodes and $D \subset N \times N$ is a binary relation on N. Such a digraph can represent various hierarchical organizations in which the arcs in D represent 'dominance relations' between the nodes in N. A digraph is called a *weak order* if it is *complete* and *transitive*. Completeness means that for every pair $i, j \in N$ it holds that $\{(i,j),(j,i)\} \cap D \neq \emptyset$. (Note that completeness implies *reflexivity*, i.e., $(i,i) \in D$ for all $i \in N$.) Transitivity means that for every triple $i, j, h \in N$ such that $(i,j) \in D$ and $(j,h) \in D$ it holds that $(i,h) \in D$. A weak order (N, D) can alternatively be represented as follows. Let $i, j \in N$. Then we denote $i \succeq j$ if and only if $(i,j) \in D$. Given this representation of a weak order we denote $i \succ j$ if and only if $(i,j) \in D$ and $(j,i) \notin D$. Further we denote $i \sim j$ if and only if $(i,j) \in D$ and $(j,i) \in D$. If $i \succeq j$ then we could say that node i is 'at least as good' as node j, while if $i \succ j$ we could say that i is 'better' than j. If $i \sim j$ then i and j are 'indifferent'. In this way a weak order can be seen as a ranking or ordering of a set of nodes. A *ranking procedure* on N is a mapping that assigns to every digraph such a ranking.

A possible interpretation of a digraph in which it is useful to rank the nodes is that it represents a competition being played among certain teams or players. In that case the nodes represent the players that participate in the competition and $(i,j) \in D$

means that player i has won the match he played against player j. This interpretation can be found in, e.g. Rubinstein (1980) and Laffont, Laslier and LeBreton (1993). Another interpretation can be found in van den Brink and Gilles (1992a). There the nodes represent a set of economic agents that are engaging in some economic trade process such that $(i, j) \in D$ means that agent i sets the conditions under which binary trade between agent j and himself will take place. (For example, agent i sets the prices under which he exchanges commodities with agent j.) Another interpretation of $(i, j) \in D$ is that agent i has *veto power* over the actions undertaken by agent j. A game theoretic analysis of this interpretation is given in Gilles, Owen and van den Brink (1992).

These are interpretations in which the set of nodes correspond to a set of agents or players. If the set of nodes represent a set of alternatives between which an agent or group of agents have to choose then the digraph (N, D) can represent a *preference relation*. Then $(i, j) \in D$ means that an agent or group of agents when pairwise comparing alternatives i and j prefers i to j (cf. Sen (1979)).

In this paper we discuss a specific ranking procedure, namely the one that is based on the *score measure*. In this ranking procedure a node is 'better' the more other nodes it dominates.

In section 2 we present the score measure and the ranking procedure that is based on it and give an axiomatic characterization of this ranking procedure. This is a generalization of the result stated in Rubinstein (1980) who axiomatized the ranking by score measure restricted to the class of *tournaments*[1].

In section 3 we extend the ranking by score measure to the class of *weighted directed graphs*. In a weighted digraph the relations are assigned weights which can be seen as measures of the 'importance' of the relations. A ranking procedure for weighted digraphs assigns a ranking of the nodes of every weighted digraph taking the weights of the relations into account. We also give an axiomatic characterization of this ranking procedure for weighted digraphs.

[1] A digraph (N, D) is a tournament if $(i, i) \notin D$ for all $i \in N$ and for all $i, j \in N$ with $i \neq j$ it holds that either $(i, j) \in D$ or $(j, i) \in D$.

We conclude this section by presenting some concepts about digraphs that will be used in this paper. We concentrate on *finite irreflexive* digraphs. A digraph (N, D) is finite if N is finite and it is irreflexive if $(i,i) \notin D$ for every $i \in N$. We simply refer to these graphs as digraphs. Note that it is allowed that both $(i,j) \in D$ and $(j,i) \in D$. In the sequel we assume the set N to be fixed and therefore we represent a digraph just by its binary relation D. The collection of all digraphs on N represented by their binary relation is denoted by $\mathcal{D}^N$.

Let $D \in \mathcal{D}^N$. If $(i,j) \in D$ then j is called a *successor* of i and i is called a *predecessor* of j in D. For $i \in N$ we define $S_D(i) := \{j \in N \mid (i,j) \in D\}$ being the collection of all successors of i in D and $P_D(i) := \{j \in N \mid (j,i) \in D\}$ being the collection of predecessors of i in D. (In the sequel we will often omit the subscript D.) A weighted digraph is a pair (N,ω) where N is a finite set of nodes and $\omega: N \times N \to \mathbb{R}_+$ is a *weight function* which assigns a non-negative real value to each ordered pair $(i,j) \in N \times N$ such that $\omega(i,i) = 0$ for all $i \in N$. The value $\omega(i,j)$ is a measure of how strongly node i dominates node j. Similarly as for (non-weighted) digraphs we represent a weighted digraph on N just by its weight function. A digraph $D \in \mathcal{D}^N$ can be seen as a weighted digraph ω which assigns the value 1 to every relation $(i,j) \in D$ and the value 0 to all other relations.

2 Ranking the nodes in a digraph

In this section we discuss a specific ranking procedure that assigns a weak order representing a ranking of the nodes to every digraph. This ranking procedure is based on the *score measure*[2].

Definition 2.1 *The* **score measure** *is the function $\sigma: \mathcal{D}^N \to \mathbb{R}^N$ given by*

$$\sigma_i(D) := \#S(i) \text{ for all } i \in N \text{ and } D \in \mathcal{D}^N.$$

[2] Although mostly used for *tournaments* the score measure can easily be generalized to the class of all digraphs as is done in Definition 2.1.

Thus the score measure assigns to each node in a digraph the number of nodes it dominates[3]. For an axiomatization of this score measure for digraphs we refer to van den Brink and Gilles (1992b). In the ranking by score measure a node is 'better' the higher its score is, i.e., the more nodes it dominates. We denote the collection of all weak orders on N by $\mathcal{WO}^N$.

Definition 2.2 *The* **ranking by score measure** *is the mapping* $\succeq_\sigma : \mathcal{D}^N \to \mathcal{WO}^N$ *which for every* $D \in \mathcal{D}^N$ *is given by*

$i \succeq_\sigma (D)\ j$ *if and only if* $\sigma_i(D) \geq \sigma_j(D)$.

In Rubinstein (1980) this ranking by score measure is axiomatized restricted to the class of tournaments. Here we generalize this result for arbitrary digraphs. The first axiom is an anonimity axiom.

Axiom 2.3 (Anonimity) *For every* $D \in \mathcal{D}^N$ *and permutation* $\pi : N \to N$ *it holds that* $i \succeq_\sigma (D)\ j$ *if and only if* $\pi(i) \succeq_\sigma (\pi D)\ \pi(j)$*, where* $\pi D \in \mathcal{D}^N$ *is given by* $(\pi(i), \pi(j)) \in \pi D$ *if and only if* $(i,j) \in D$.

The second axiom states that given a particular digraph it is always better to have more successors.

Axiom 2.4 (Positive responsiveness) *Let* $D \in \mathcal{D}^N$ *and* $i, j, h \in N$, $i \neq j$ *be such that* $(i,h) \notin D$. *Further, let* $D' = D \cup \{(i,h)\}$.
If $i \succeq (D)\ j$ *then* $i \succ (D')\ j$.

Finally, the third axiom states that the order between two nodes does not change if changes only take place in relations on which they are not the dominating nodes.

Axiom 2.5 (Independence of non-dominated relations) *Let* $D, D' \in \mathcal{D}^N$ *and* $i, j \in N$ *be such that* $S_D(i) = S_{D'}(i)$ *and* $S_D(j) = S_{D'}(j)$. *Then*

$i \succeq (D)\ j$ *if and only if* $i \succeq (D')\ j$.

[3] For some properties of the score measure for tournaments we refer to Behzad, Chartrand and Lesniak-Foster (1979).

These three axioms uniquely determine the ranking by score measure. Rubinstein (1980) also uses anonimity and positive responsiveness in axiomatizing this ranking procedure restricted to the class of tournaments. The independence of non-dominated relations axiom is a strengthening of the corresponding axiom used by Rubinstein which states that the order between two nodes does not change if changes only take place in relations on which they are neither the predecessor nor the successor.

Theorem 2.6 *The ranking procedure* $\succeq: \mathcal{D}^N \to \mathcal{WO}^N$ *is equal to ranking by score measure if and only if it satisfies anonimity, positive repsponsiveness and independence of non-dominated relations.*

PROOF

It is easy to check that the ranking by score measure satisfies the three axioms.

Now suppose that the ranking procedure $\succeq: \mathcal{D}^N \to \mathcal{WO}^N$ satisfies the three axioms and let $D \in \mathcal{D}^N$. We prove that it has to be the ranking by score measure in three steps.

(a)

We first prove that for each pair $i, j \in N$ it holds that $i \sim(D)\ j$ if $\sigma_i(D) = \sigma_j(D)$.

Therefore let $i, j \in N$ be such that $\sigma_i(D) = \sigma_j(D)$.

Consider the digraph $\widehat{D}$ where $\widehat{D} := \{(i,h) \mid h \in S(i)\} \cup \{(j,h) \mid h \in S(j)\}$. For this digraph it holds that $S_D(i) = S_{\widehat{D}}(i)$ and $S_D(j) = S_{\widehat{D}}(j)$. We now prove that $i \sim(\widehat{D})\ j$. (It is clear that independence of non-dominated relations then implies that $i \sim(D)\ j$.)

We distinguish the following four cases with respect to the pair i, j.

(i) Suppose that $(i,j) \notin D$ and $(j,i) \notin D$.

Since $\sigma_i(D) = \sigma_j(D)$ and $i \notin [S_{\widehat{D}}(j) \cup P_{\widehat{D}}(j)]$ it follows from anonimity of $\succeq$ that $i \sim(\widehat{D})\ j$.

The digraph $\widehat{D}$ in this case is illustrated in Figure 1. In this figure $A = S_D(i) \setminus S_D(j)$, $B = S_D(j) \setminus S_D(i)$, $C = S_D(i) \cap S_D(j)$, and $E = N \setminus [S_D(i) \cup S_D(j) \cup \{i,j\}]$. It is clear that $\#A = \#B$ since $\sigma_i(\widehat{D}) = \sigma_j(\widehat{D})$.

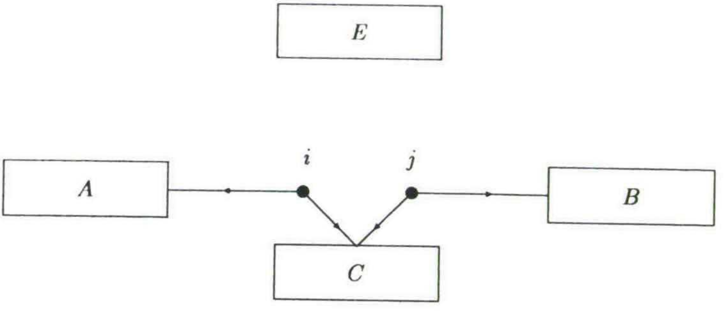


Figure 1: Digraph $\widehat{D}$ in case $(i,j) \notin D$ and $(j,i) \notin D$

(ii) Suppose that $(i,j) \in D$ and $(j,i) \in D$.

Since $\sigma_i(D) = \sigma_j(D)$ and $i \in [S_{\widehat{D}}(j) \cap P_{\widehat{D}}(j)]$ it follows from anonimity of $\succeq$ that $i \sim(\widehat{D})\ j$.

The digraph $\widehat{D}$ in this case is illustrated in Figure 2. In this figure $A = S_D(i) \setminus [S_D(j) \cup \{j\}]$, $B = S_D(j) \setminus [S_D(i) \cup \{i\}]$, $C = S_D(i) \cap S_D(j)$, and $E = N \setminus [S_D(i) \cup S_D(j) \cup \{i,j\}]$. Again $\#A = \#B$.

(iii) Suppose that $(i,j) \in D$ and $(j,i) \notin D$.

Since $\sigma_i(D) = \sigma_j(D)$ it must hold that $S_{\widehat{D}}(j) \backslash S_{\widehat{D}}(i) \neq \emptyset$. Let $h \in S_{\widehat{D}}(j) \backslash S_{\widehat{D}}(i)$. Let $A = S_D(i) \setminus (S_D(j) \cup \{j\})$, $B = S_D(j) \setminus (S_D(i) \cup \{h\})$, $C = S_D(i) \cap S_D(j)$, and $E = N \setminus (S_D(i) \cup S_D(j) \cup \{i,j\})$. (Note that $\#A = \#B$.) The digraph $\widehat{D}$ in this case is illustrated in Figure 3.

Let $A = \{a_1, \ldots, a_t\}$ and $B = \{b_1, \ldots, b_t\}$. Next let $D' \in \mathcal{D}^N$ be given by

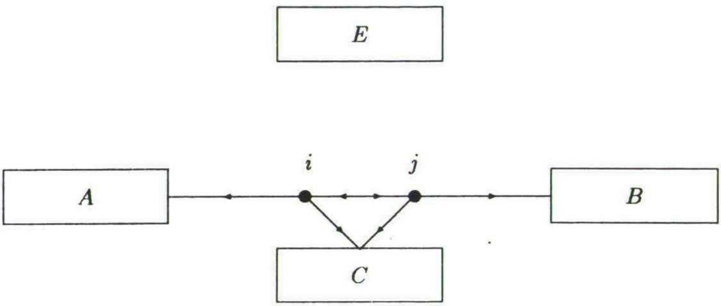


Figure 2: Digraph $\widehat{D}$ in case $(i,j) \in D$ and $(j,i) \in D$

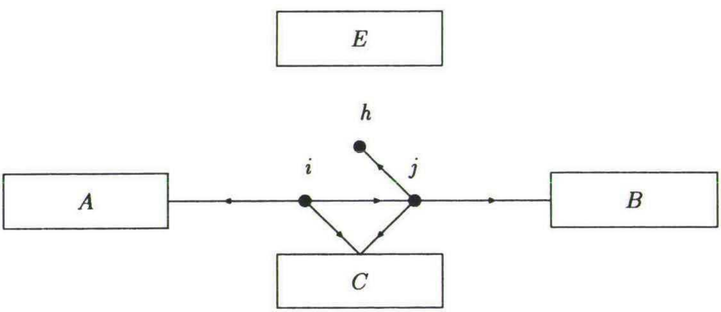


Figure 3: Digraph $\widehat{D}$ in case $(i,j) \in D$, $(j,i) \notin D$

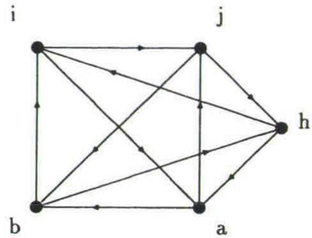


Figure 4: Digraph D'

$$D' = \widehat{D} \cup \{(h, a_k) \mid 1 \leq k \leq t\} \cup \{(h, i)\} \cup \{(h, g) \mid g \in C\}$$

$$\cup \Big(\cup_{k=1}^{t} (\{(a_k, b_l) \mid k \leq l \leq t\} \cup \{(a_k, a_l) \mid 1 \leq l \leq k-1\} \cup \{(a_k, j)\} \cup \{(a_k, g) \mid g \in C\})\Big)$$

$$\cup \Big(\cup_{k=1}^{t} (\{(b_k, a_l \mid k+1 \leq l \leq t\} \cup \{(b_k, b_l) \mid 1 \leq l \leq k-1\} \cup \{(b_k, i), (b_k, h)\} \cup \{(b_k, g) \mid g \in C\})\Big)$$

Digraph D' is illustrated in Figure 4 in case $\#A = \#B = 1$ (in this figure we have deleted the nodes in C and E).

From anonimity of $\succeq$ it follows that $i \sim (D')\ j$.

Since $S_{D'}(i) = S_{\widehat{D}}(i)$ and $S_{D'}(j) = S_{\widehat{D}}(j)$ it follows from independence of non-dominated relations that $i \sim (\widehat{D})\ j$.

(iv) Suppose that $(i, j) \notin D$ and $(j, i) \in D$.

Then $i \sim (\widehat{D})\ j$ follows similarly as under (iii) with the roles of i and j reversed.

Since $i \sim (\widehat{D})\ j$ as shown above, $S_D(i) = S_{\widehat{D}}(i)$ and $S_D(j) = S_{\widehat{D}}(j)$ it follows with independence of non-dominated relations that $i \sim (D)\ j$.
Thus we have proved that under anonimity and independence of non-dominated relations it holds that $i \sim (D)\ j$ if $\sigma_i(D) = \sigma_j(D)$.

(b)

Now suppose without loss of generality that $\sigma_i(D) > \sigma_j(D)$.
Let the digraph $\widetilde{D}$ be such that the following conditions are satisfied:

- $\widetilde{D} \subset D$;
- For every $h \in N \setminus \{i\}$ and every $g \in N$ it holds that $(h, g) \in \widetilde{D}$ if and only if $(h, g) \in D$;
- $\sigma_i(\widetilde{D}) = \sigma_j(\widetilde{D})$.

(Since $\sigma_i(D) > \sigma_j(D)$ such a $\widetilde{D}$ always exists.)
As shown above it follows from anonimity and independece of non-dominated relations that $i \sim(\widetilde{D})\ j$.
Repeated application of the positive responsiveness axiom then yields that $i \succ(D)\ j$.
Thus $i \succeq(D)\ j$ if $\sigma_i(D) \geq \sigma_j(D)$.

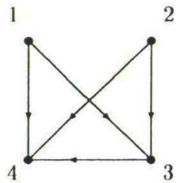
(c)
Under (a) we have shown that $j \succ(D)\ i$ if $\sigma_i(D) < \sigma_j(D)$. This implies that $\sigma_i(D) \geq \sigma_j(D)$ if $i \succeq(D)\ j$.

Thus we have proved that if a ranking procedure satisfies the three axioms then it has to be the ranking by score measure.

□

We conclude this section by giving an example which illustrates that all three axioms that are stated in Theorem 2.6 are necessary in order to uniquely determine the ranking by score measure.

Example 2.7 Consider the digraph D on $N = \{1, 2, 3, 4\}$ which is given by $D = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.



The score measure of this digraph is given by $(2,2,1,0)$.
Thus the ranking by score measure yields the order: $1 \sim_\sigma 2 \succ_\sigma 3 \succ_\sigma 4$.
Next we give four alternative ranking procedures that each satisfy two but not all three of the axioms stated in Theorem 2.6.

1. Let the nodes in N be labelled by the numbers 1 to n and let the ranking procedure $\succeq_1$ for every $D \in \mathcal{D}^N$ be given by

   $i \succ_1 (D)\ j$ if and only if either $[\#S(i) > \#S(j)]$

   or $[\#S(i) = \#S(j)$ and $i < j]$

   This ranking procedure satisfies all three axioms except anonimity. For the digraph D given above it holds that $1 \succ_1 (D)\ 2$ although $S(1) = S(2)$ and $P(1) = P(2)$.

2. Let the ranking procedure $\succeq_2$ be such that all nodes are 'equal' irrespective of the dominance relation, i.e., for every $D \in \mathcal{D}^N$ it holds that $i \sim_2 (D)\ j$ for all $i, j \in N$.

   This ranking procedure satisfies all three axioms except positive responsiveness. Consider the digraph D given above and the digraph $D' = D \cup \{(1,2)\}$.

   If the positive responsiveness axiom is satisfied then it must hold that $1 \succ_2 (D')\ 2$ since $1 \sim_2 (D)\ 2$. But $1 \sim_2 (D')\ 2$.

3. Let the ranking procedure $\succeq_3$ for every $D \in \mathcal{D}^N$ be given by

$$i \succeq_3(D)\, j \text{ if and only if } \#S(i) - \#P(i) \geq \#S(j) - \#P(j).$$

This ranking procedure satisfies all three axioms except independence of non-dominated relations. Consider the digraph D given above and the digraph $D' = D \cup \{(3,2)\}$.

If the independence of non-dominated realtions axiom is satisfied then it must hold that $1 \sim_3(D')\, 2$ since $1 \sim_3(D)\, 2$ and D' is as described in axiom 2.5. But $1 \succ_3(D')\, 2$.

3 Weighted directed graphs

A dominance structure on a set of nodes N in which not all dominance relations are equally important can be represented by a *weighted directed graph* (N, ω) where the weight function $\omega: N \times N \to \mathbb{R}_+$ is such that $\omega(i,i) = 0$ for all $i \in N$. The value $\omega(i,j)$ is a measure of how strongly node i dominates node j. If $\omega(i,j) = 0$ then node i does not dominate node j at all. In this section we generalize the score measure to the class of weighted digraphs and give axiomatizations of the ranking procedure that is based on it. We denote the collection of all weighted digraphs on N represented by their weight function by $\mathcal{W}^N$.

Definition 3.1 *The* **weighted score measure** *is the function* $\sigma: \mathcal{W}^N \to \mathbb{R}^N$ *which is given by*

$$\sigma_i(\omega) := \sum_{j \in N} \omega(i,j) \textit{ for all } i \in N \textit{ and } \omega \in \mathcal{W}^N.$$

Thus the weighted score measure assigns to every node $i \in N$ in a weighted digraph ω the sum of the weights of all relations on which i is the dominating node. As said in the introduction, a (non weighted) digraph D can be seen as a weighted digraph ω with

$$\omega(i,j) = \begin{cases} 1 & \text{for all } (i,j) \in D \\ 0 & \text{else.} \end{cases}$$

Then it is easy to see that $\sigma_i(\omega)$ is equal to the score of i in the (non-weighted) digraph D. Thus the weighted score measure indeed is a generalization of the score measure. Next we straightforwardly generalize the ranking procedure for weighted digraphs that is based on this weighted score measure.

Definition 3.2 *The* **ranking by weighted score measure** *is the mapping* $\succeq_\sigma : \mathcal{W}^N \to \mathcal{WO}^N$ *which for every* $\omega \in \mathcal{W}^N$ *is given by*

$i \succeq_\sigma (\omega)\ j$ *if and only if* $\sigma_i(\omega) \geq \sigma_j(\omega)$.

The ranking by weighted score measure can be axiomatized by generalizing the three axioms that axiomatized the ranking by score measure in Theorem 2.6 plus adding a new axiom.

Axiom 3.3 (Weighted anonimity) *For every* $\omega \in \mathcal{W}^N$ *and permutation* $\pi: N \to N$ *it holds that* $i \succeq_\sigma (\omega)\ j$ *if and only if* $\pi(i) \succeq_\sigma (\pi\omega)\ \pi(j)$, *where* $\pi\omega \in \mathcal{W}^N$ *is given by* $\pi\omega(\pi(i), \pi(j)) = \omega(i,j)$ *for all* $(i,j) \in N \times N$.

Axiom 3.4 (Weighted positive responsiveness) *Let* $\omega \in \mathcal{W}^N$ *and let* $\omega' \in \mathcal{W}^N$ *be such that for some pair* $i, h \in N$ *there is a positive constant* $c > 0$ *such that:*

$$\omega'(p,q) = \begin{cases} \omega(p,q) + c & \text{if } (p,q) = (i,h) \\ \\ \omega(p,q) & \text{else.} \end{cases}$$

If $i \succeq (\omega)\ j$ *then* $i \succ (\omega')\ j$.

Axiom 3.5 (Independence of weighted non-dominated relations) *Let* $\omega, \omega' \in \mathcal{W}^N$ *and* $i, j \in N$, $i \neq j$ *be such that* $\omega'(i,h) = \omega(i,h)$ *and* $\omega'(j,h) = \omega(j,h)$ *for all* $h \in N$. *Then*

$i \succeq (\omega)\ j$ *if and only if* $i \succeq (\omega')\ j$.

Let $\omega, \omega' \in \mathcal{W}^N$. Then we define $(\omega + \omega') \in \mathcal{W}^N$ by $(\omega + \omega')(i,j) := \omega(i,j) + \omega'(i,j)$ for all $(i,j) \in N \times N$. Next we introduce a new axiom which states that if we add two weighted digraphs in the way described above and node i is at least as good as node j in both weighted digraphs then node i is at least as good as node j in the sum digraph.

Axiom 3.6 (Order preservation) *Let $\omega, \omega' \in \mathcal{W}^N$ and let $i, j \in N$. If $i \succeq(\omega)\, j$ and $i \succeq(\omega')\, j$ then $i \succeq(\omega + \omega')\, j$.*

These four axioms uniquely determine the ranking by weighted score measure for weighted digraphs.

Theorem 3.7 *The ranking procedure $\succeq: \mathcal{W}^N \to \mathcal{WO}^N$ is equal to the ranking by weighted score measure if and only if it satisfies weighted anonimity, weighted positive responsiveness, independence of weighted non-dominated relations and order preservation.*

PROOF

It is easy to check that the ranking by weighted score measure satisfies the four axioms. Now suppose that the ranking procedure $\succeq: \mathcal{W}^N \to \mathcal{WO}^N$ satisfies the four axioms and let $\omega \in \mathcal{W}^N$.

(a)

We first prove that for each pair $i, j \in N$ it holds that $i \sim(\omega)\, j$ if $\sigma_i(\omega) = \sigma_j(\omega)$.
Therefore let $i, j \in N$ be such that $\sigma_i(\omega) = \sigma_j(\omega)$.
Consider the weighted digraph $\widehat{\omega} \in \mathcal{W}^N$ given by

$$\widehat{\omega}(h,g) = \begin{cases} \omega(h,g) & \text{if } h \in \{i,j\},\ g \in N \\ 0 & \text{else.} \end{cases}$$

Further for every $\omega \in \mathcal{W}^N$ we introduce the following.

$$m(\omega) := \min\{\omega(h,g) \mid (h,g) \in N \times N \text{ and } \omega(h,g) > 0\}$$

$$M(\omega) := \{(h,g) \in N \times N \mid \omega(h,g) = m(\omega)\}$$

We may suppose without loss of generality that $\omega(i,j) \geq \omega(j,i)$.
Next we construct a collection of weighted digraphs $\{\omega^k\}_{0\leq k\leq t}$ for some finite $t \in \mathbb{N}$ such that nodes i and j are 'symmetric' in each of these digraphs, and thus we can apply weighted anonimity to conclude that $i \sim (\omega^k)\ j$ in all those digraphs. We construct this collection of weighted digraphs using the following procedure.

STEP 1 Let ω^0 be given by

$$\omega^0(h,g) = \begin{cases} \omega(j,i) & \text{if } (h,g) \in \{(i,j),(j,i)\} \\ 0 & \text{else.} \end{cases}$$

(Anonimity of $\succeq$ clearly implies that $i \sim(\omega^0)\ j$.)

Now let $\overline{\omega}^1 \in \mathcal{W}^N$ be given by

$$\overline{\omega}^1(h,g) := \begin{cases} \widehat{\omega}(h,g) - \omega(j,i) & \text{if } (h,g) \in \{(i,j),(j,i)\} \\ \widehat{\omega}(h,g) & \text{else.} \end{cases}$$

(Note that $\overline{\omega}^1(j,i) = 0$ and $\overline{\omega}^1(i,j) \geq 0$.)

Let $k = 0$.

STEP 2 IF $\{(h,g) \in N \times N \mid \overline{\omega}^{k+1}(h,g) > 0\} = \emptyset$ then let $t = k$ and STOP.

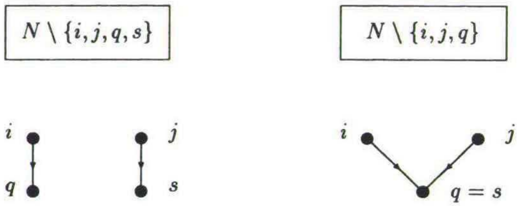
ELSE since $\sigma_i(\widehat{\omega}) = \sigma_j(\widehat{\omega})$ and $\sigma_i(\omega^m) = \sigma_j(\omega^m)$ for all $0 \leq m \leq k$, we know that there must exist at least one $h \in N \setminus \{i\}$ such that $\overline{\omega}^{k+1}(i,h) > 0$ and at least one $g \in N \setminus \{i,j\}$ such that $\overline{\omega}^{k+1}(j,g) > 0$.

Let $k = k+1$ and GOTO STEP 3.

STEP 3 Take a $(p,q) \in M(\overline{\omega}^k)$. (Note that $p \in \{i,j\}$.) Since $\sigma_i(\widehat{\omega}) = \sigma_j(\widehat{\omega})$ and $\sigma_i(\omega^m) = \sigma_j(\omega^m)$ for all $0 \leq m < k$ there exists an $(r,s) \in N \times N$ such that $r = \{i,j\} \setminus \{p\}$ and $\omega(r,s) > 0$.

Let $\overline{\omega}^{k+1}\colon N \times N \to \mathbb{R}$ be given by

$$\overline{\omega}^{k+1}(h,g) = \begin{cases} \overline{\omega}^k(h,g) - m(\overline{\omega}^k) & \text{if } (h,g) \in \{(p,q),(r,s)\} \\ \overline{\omega}^k(h,g) & \text{else.} \end{cases}$$



$\omega^k(i,q) = \omega^k(j,s)$ $\qquad$ $\omega^k(i,q) = \omega^k(j,q)$

Figure 5: Weighted digraph ω^k in case $(i,j) \notin \{(p,q),(r,s)\}$

(Clearly $\overline{\omega}^{k+1}(p,q) = 0$ and $\overline{\omega}^{k+1}(h,g) \geq 0$ for all $(h,g) \neq (p,q)$, thus $\omega^{k+1} \in \mathcal{W}^N$.)

Next we distinguish between the following two cases:

IF $(i,j) \notin \{(p,q),(r,s)\}$ then let ω^k be given by

$$\omega^k(h,g) := \begin{cases} m(\overline{\omega}^k) & \text{if } (h,g) \in \{(p,q),(r,s)\} \\ 0 & \text{else.} \end{cases}$$

The weighted digraph ω^k is illustrated in Figure 5 (with $p = i$ and $r = j$) in case $q \neq s$ respectively if $q = s$. (Note that anonimity implies that $i \sim(\omega^k)\, j$.)

GOTO STEP 2.

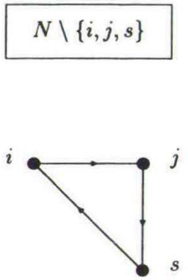
ELSE $(i,j) \in \{(p,q),(r,s)\}$. Suppose that $(i,j) = (p,q)$ (and thus $r = j$). Then let ω^k be given by

$$\omega^k(h,g) := \begin{cases} m(\overline{\omega}^k) & \text{if } (h,g) \in \{(i,j),(j,s),(s,i)\} \\ 0 & \text{else.} \end{cases}$$

The weighted digraph ω^k is illustrated in Figure 6.

(Anonimity also implies that $i \sim(\omega^k)\, j$ in this case.)

If $(i,j) = (r,s)$ then we do the same but with s replaced by q. GOTO STEP 2.



$$\omega^k(i,j) = \omega^k(j,s) = \omega^k(s,i)$$

Figure 6: Weighted digraph ω^k in case $(i,j) \in \{(p,q),(r,s)\}$

Since $\sigma_i(\widehat{\omega}) = \sigma_j(\widehat{\omega})$ this procedure leads to a collection of weighted digraphs $\{\omega^k\}_{0 \leq k \leq t}$ which have been constructed such that nodes i and j are 'symmetric' in each of these weighted digraphs.
Weighted anonimity then implies that for every ω^k, $0 \leq k \leq t$, it holds that $i \sim(\omega^k)\ j$.

Now let $\omega' \in \mathcal{W}^N$ be given by $\omega'(i,j) := \sum_{k=0}^{t} \omega^k(i,j)$ for all $(i,j) \in N \times N$.
Order preservation then implies that $i \sim(\omega')\ j$.

If $\omega(i,j) = \omega(j,i)$ then $\omega^0(i,j) = \omega^0(j,i) = \omega(i,j)$ and $\overline{\omega}^k(i,j) = \overline{\omega}^k(j,i) = 0$ for all $1 \leq k \leq t$ and thus case 2 in step 3 cannot occur. But then $\omega' = \widehat{\omega}$ and thus $i \sim(\widehat{\omega})\ j$.
Else $\omega(i,j) > \omega(j,i)$, and then there is some $s \in N \setminus \{i\}$ and some positive constant $c > 0$ such that

$$\omega'(h,g) = \begin{cases} \widehat{\omega}(h,g) + c & \text{if } (h,g) = (s,i) \\ \widehat{\omega}(h,g) & \text{else.} \end{cases}$$

(This s is the one in Figure 6.)
Independence of non-dominated weighted relations then yields that in this case also $i \sim(\widehat{\omega})\ j$.
Thus $i \sim(\widehat{\omega})\ j$, and since $\omega(i,g) = \widehat{\omega}(i,g)$ and $\omega(j,g) = \widehat{\omega}(j,g)$ for all $g \in N$ independence of non-dominated weighted relations implies that $i \sim(\omega)\ j$.

Thus we have shown that $i \sim(\omega)\ j$ if $\sigma_i(\omega) = \sigma_j(\omega)$.

(b)

Next suppose without loss of generality that $\sigma_i(\omega) > \sigma_j(\omega)$.
Let the weighted digraph $\tilde{\omega} \in \mathcal{W}^N$ be such that the following conditions are satisfied:

- $\tilde{\omega}(h,g) \leq \omega(h,g)$ for all $(h,g) \in N \times N$;
- $\tilde{\omega}(h,g) = \omega(h,g)$ for all $h \in N \setminus \{i\}$ and $g \in N$;
- $\sigma_i(\tilde{\omega}) = \sigma_j(\tilde{\omega})$.

Since $\sigma_i(\omega) > \sigma_j(\omega)$ such a $\tilde{\omega}$ always exists.
As shown above it follows from weighted anonimity, independence of non-dominated weighted relations, and the order preserving property that $i \sim(\tilde{\omega})\ j$.
Repeated application of the weighted positive responsiveness axiom then yields that $i \succ(\omega)\ j$.
Thus $i \succeq(\omega)\ j$ if $\sigma_i(\omega) \geq \sigma_j(\omega)$.

(c)

We have shown above that $j \succ(\omega)\ i$ if $\sigma_i(\omega) < \sigma_j(\omega)$. This implies that $\sigma_i(\omega) \geq \sigma_j(\omega)$ if $i \succeq(\omega)\ j$.

Thus we have proved that if a ranking procedure satisfies the four axioms stated in Theorem 3.7 then it has to be the ranking by weighted score measure.

□

We conclude this paper by giving an example which illustrates that all four axioms that are stated in Theorem 3.7 are necessary in order to uniquely determine the ranking by weighted score measure for the class of weighted digraphs.

Example 3.8 Consider the weighted digraph ω on $N = \{1, \ldots, 4\}$ which is given by:

$$\omega(i,j) = \begin{cases} 1 & \text{if } (i,j) \in \{(1,3),(1,4),(2,3),(2,4)\} \\ 2 & \text{if } (i,j) = (3,4) \\ 0 & \text{else.} \end{cases}$$

Then $\sigma(\omega) = (2,2,2,0)$ and the ranking by generalized score measure yields: $1 \sim_\sigma (\omega)\ 2 \sim_\sigma (\omega)\ 3 \succ_\sigma (\omega)\ 4$. Next we give four alternative ranking procedures for weighted digraphs that each satisfy three but not all four of the axioms stated in Theorem 3.7. The first three alternative ranking procedures are generalizations of the three ranking procedures that are given in Example 2.7.

1. Let the nodes in N be labelled by the numbers 1 to n and let the ranking procedure $\succeq_4$ for every $\omega \in \mathcal{W}^N$ be given by

   $$i \succ_4 (\omega)\ j \quad \text{if and only if} \quad \text{either } [\sigma_i(\omega) > \sigma_j(\omega)]$$

   $$\text{or } [\sigma_i(\omega) = \sigma_j(\omega) \text{ and } i < j]$$

   Note that $i \succeq_4 (\omega)\ j$ if and only if $i \succ_4 (\omega)\ j$. This ranking procedure satisfies all four axioms except weighted anonimity. For the weighted digraph ω given above it holds that $1 \succ_4 (\omega)\ 2$ although $\omega(1,h) = \omega(2,h)$ and $\omega(h,1) = \omega(h,2)$ for all $h \in N$.

2. Let the ranking procedure $\succeq_5$ for every $\omega \in \mathcal{W}^N$ be given by: $i \sim_5 (\omega)\ j$ for all $i,j \in N$.

   This ranking procedure satisfies all four axioms except weighted positive responsiveness. Consider the weighted digraph ω given above and the weighted digraph $\omega' = (N, \omega')$ where $\omega'(i,j) = \begin{cases} 2 & \text{if } (i,j) = (1,3) \\ \omega(i,j) & \text{else.} \end{cases}$

   If the weighted positive responsiveness axiom is satisfied then it must hold that $1 \succ_5 (\omega')\ 2$ since $1 \sim_5 (\omega)\ 2$. But $1 \sim_5 (\omega')\ 2$.

3. Let the ranking procedure $\succeq_6$ for every $\omega \in \mathcal{W}^N$ be given by

   $$i \succeq_6 (D)\ j \quad \text{if and only if} \quad \sum_{h \in N} \omega(i,h) - \sum_{h \in N} \omega(h,i) \geq$$

   $$\sum_{h \in N} \omega(j,h) - \sum_{h \in N} \omega(h,j)$$

This ranking procedure satisfies all four axioms except independence of non-dominated weighted relations. Consider the weighted digraph ω given above and the weighted digraph ω' where $\omega'(i,j) = \begin{cases} 1 & \text{if } (i,j) = \{(3,2)\} \\ \omega(i,j) & \text{else.} \end{cases}$

If independence of non-dominated weighted relations is satisfied then it must hold that $1 \sim_6 (\omega')\ 2$ since $1 \sim_6 (\omega)\ 2$ and ω' is as described in axiom 3.5. But $1 \succ_6 (\omega')\ 2$.

4. Let $\overline{\sigma}\colon \mathcal{W}^N \to \mathbb{R}^N$ be given by:

$$\overline{\sigma}_i(\omega) := \#\{j \in N \mid \omega(i,j) > 0\} \text{ for all } i \in N \text{ and } \omega \in \mathcal{W}^N.$$

Thus $\overline{\sigma}$ assigns to every weighted digraph ω the score measure of the (non-weighted) digraph D where $D = \{(i,j) \in N \times N \mid \omega(i,j) > 0\}$. Now let the ranking procedure $\succeq_7$ for every $\omega \in \mathcal{W}^N$ be given by

$$i \succeq_7 (\omega)\ j \text{ if and only if } \overline{\sigma}_i(\omega) \cdot \sigma_i(\omega) \geq \overline{\sigma}_j(\omega) \cdot \sigma_j(\omega).$$

This ranking procedure satisfies all four axioms except the order preserving property. Consider the weighted digraphs ω^1 and ω^2, where

$$\omega^1(i,j) = \begin{cases} 1 & \text{for all } (i,j) \in \{(1,3),(2,3),(2,4),(3,4)\} \\ 0 & \text{else,} \end{cases}$$

and

$$\omega^2(i,j) = \begin{cases} 1 & \text{for all } (i,j) \in \{(1,4),(3,4)\} \\ 0 & \text{else.} \end{cases}$$

Then $3 \succeq_7 (\omega^1)\ 1$ and $3 \succeq_7 (\omega^2)\ 1$. Since $(\omega^1 + \omega^2) = \omega$, if the order preserving property is satisfied it must hold that $3 \succeq_7 (\omega)\ 1$. But $1 \succ_7 (\omega)\ 3$.

Thus all four axioms that are stated in Theorem 3.7 are necessary in order to uniquely determine the ranking by generalized score measure for weighted digraphs.

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